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Only a Dictatorship is Efficient or Neutral

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Abstract

In many, if not most, elections, several different *seats* must be filled, so that a group of candidates, or an *assembly*, is selected. Typically in these elections, voters cast their ballots on a seat-by-seat basis. We show that *seat-by-seat* procedures are efficient or neutral only under extreme conditions.

How should a voting system be judged? A time-honoured approach judges a system on the basis of the properties it satisfies. The literature on voting has considered myriad properties, including anonymity, neutrality, efficiency, independence of irrelevant alternatives, monotonicity, and Condorcet consistency. Although Arrow's impossibility theorem (1963) famously warned that a given property may be more subtle and difficult to satisfy than is initially apparent, some of these properties are generally taken to be obviously desirable and easily satisfied, both in theory and in practice. Three such properties are efficiency, anonymity, and neutrality. After all, efficiency merely requires that, when all voters prefer outcome A to outcome B, outcome B not be chosen, while anonymity and neutrality only ask, respectively, that all voters and all outcomes be treated equally. Nevertheless, we argue in this paper that, while anonymity is a pervasive feature of political voting systems, virtually no system found in practice is either efficient or neutral.

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Nor are either of these features easily obtained. The failure to fully appreciate these facts reveals that, to a large extent, political elections have not been properly analyzed.

In social choice theory, a voting rule is conceived of as a mapping from preferences over possible outcomes to a specific choice (or choices). Actual election procedures, however, do not have this structure, or, more precisely, have a very restricted structure. In a typical election – be it for a city government, a school board, or a national congress – several people, or an *assembly*, are elected. However, although the outcome is an assembly, in practice voters are not asked to vote for assemblies qua assemblies; rather they cast their votes for *individual* candidates and these candidates have their votes tallied as individuals.[1] This divergence has important consequences.

Consider, for instance, a local election for sheriff, judge, and fire chief, and suppose that two candidates present themselves for each post. A common election procedure has each seat decided by a plurality election. Plurality is, of course, an efficient method, when a *single* candidate is being chosen. However, a single candidate is not being chosen here; rather, a three-person assembly is.

One potential difficulty, which will not concern us, is that there may be perceived complementarities among the candidates. For instance, a particular voter may like sheriff candidate A_S, but only if A_S's natural inclinations are tempered by the presence of judge A_j; without A_J's presence she feels that A_S would be a terrible sheriff. Does she like or dislike A_S? It is unclear, and it is unclear how she should vote. When interdependencies exist, it is unsurprising for an inefficient assembly to be elected. We bypass this well-recognized problem and restrict our attention to the "good" case, where interdependencies are not present, so that, if, say, a voter prefers sheriff candidate A_S to candidate B_S, then he or she prefers A_S regardless of the judge and fire chief who accompany her.[2] Each voter then has well-defined rankings of the candidates for each seat. The following example shows that, even in this good case, plurality rule may be inefficient.

Suppose there are three voters, with seat preferences as given in the chart below:

[1] In some systems, citizens vote for party lists, which then form part of parliament. The party lists can be interpreted as individuals, and the parliament as the assembly.

[2] Preferences are then said to be "separable." This notion is defined formally in Section 1.

Example 1.

Preferences for Sheriff

Voter 1	Voter 2	Voter 3
B_S	A_S	A_S
A_S	B_S	B_S

Preferences for Judge

Voter 1	Voter 2	Voter 3
A_J	B_J	A_J
B_J	A_J	B_J

Preferences for Fire Chief

Voter 1	Voter 2	Voter 3
A_F	A_F	B_F
B_F	B_F	A_F

When each voter votes for his preferred candidate for each seat, the resulting assembly is $(A_S A_J A_F)$. Furthermore, this assembly seems to have unusually strong support. Indeed, each voter has voted for two thirds of the assembly. At the same time, each elected candidate has received two thirds of the vote. These statistics, however, are misleading. Suppose that Voter 1's primary concern is to have his favorite sheriff elected, so that he prefers any assembly with B_S to any assembly without B_S. Similarly, suppose that Voter 2's primary concern is with her favorite judge, and Voter 3's with his favorite fire chief. Then all voters will prefer the assembly $(B_S B_J B_F)$ to the elected assembly $(A_S A_J A_F)$. This inefficiency is not specific to plurality voting. On the contrary, we establish an impossibility result: when voting is done on a seat-by-seat basis, the *only* voting system that is efficient is a dictatorship. As to neutrality, the concept requires some care in defining properly, but we will argue that a dictatorship is also the only system that is neutral.

This work continues a line of inquiry we began with Benoît and Kornhauser (1991, 1994, 1995, 1999). In that work, we extend the concept of sincere voting to candidate-based elections. We argue that when agents vote indirectly for assemblies, the two ideas of sincerity – truthful revelation of preferences and non-strategic action – come apart. We define *simple* voting in terms of the second idea of non-strategic action. We then establish a limited inefficiency result: constant scoring systems in at-large elections are inefficient, even when preferences are separable.[3] At the same time, we

[3] A constant scoring system is one in which each voter casts k votes for k different

identify a (strong) restriction on preferences that ensures efficiency. Finally, we show that, when assembly preferences derive from more basic preferences over legislative outcomes, they will be separable only under severe conditions.

With two candidates per seat, the inefficiency of plurality rule in designated seat elections is formally equivalent to the Ostrogorsky paradox on issue-by-issue voting (Anscombe 1976, Bezembinder and Acker 1985, Daudt and Rae 1976, Deb and Kelsey 1987). Oskal-Sanver and Sanver (2006) further develop this two-candidate framework. Their work adopts the interpretation of referendum voting and builds on the "paradox" noted in Brams et al. (1998).[4] In our terms, they prove that no anonymous seat-based procedure with exactly two candidates per seat and at least three seats is inefficient.[5] Our Theorem 1 generalizes their result in at least three respects. First, and most importantly, our theorem shows that dropping anonymity is of virtually no help. Second, our theorem covers the case of only two seats. Finally, our inefficiency result holds when there are more than two candidates for a seat.[6] As far as we know, our result on neutrality – Theorem 2 – has no parallel.

Our discussion proceeds as follows. In the next section, we set out the basic concepts. In section 2, we set out the results for designated seat assemblies. In section 3, we extend our results to many common election procedures for at-large assemblies. In Section 4, we discuss the intuition behind, and the implications of, our results. Proofs appear in the appendix.

1 Basic Concepts

Election procedures are remarkably varied. We impose some order on this variety by classifying procedures for electing assemblies according to whether or not candidates must declare which seat they contest. An assembly in which

candidates.

[4] Our formulation is easily interpreted as a model of referenda: Each referendum is a seat contested by two candidates, for and against.

[5] Actually, they prove something somewhat weaker, as they restrict each seat to being determined by the same voting rule.

[6] The methodology of Oskal-Sanver and Sanver relies crucially on the fact that there are only two options per seat. In particular, their Theorem 3.1 is not true when there are more than two options. Nevertheless, their main inefficiency result – Theorem 3.2 – is easily extended to the case of more than two options per seat, so that it is fair to say that this theorem is more general than its statement indicates. On the other hand, Theorem 3.4 does not extend.

candidates must declare the seat they contest is a *designated-seat* assembly. Assemblies in which candidates do not declare which seat they contest are *at-large* assemblies. For the most part, we concentrate our attention in this paper on designated-seat elections, and we develop the formalism in this section for this type of election.

Let $N = \{1, ..., n\}$ be the set of voters, let $S = \{1, ..., s\}$ be the seats contested, and let $\mathbb{C}_i$, $i = 1, 2, ..., s$ be the candidates contesting seat i. An assembly $\mathcal{A}$ is an element of $\mathbb{A} = \mathbb{C}_1 \times \cdots \times \mathbb{C}_s$. Let $\mathbb{L}$ be the set of linear orders over $\mathbb{A}$, and let $\mathbb{L}^n = \mathbb{L} \times \cdots \times \mathbb{L}$ (n times). For $L \in \mathbb{L}$, let $\mathcal{A} \succ_L \mathcal{B}$ mean that $\mathcal{A}$ is ranked higher than $\mathcal{B}$ according to L.

Since the choice problem at hand is the selection of an assembly, social choice theory takes individual rankings of the assemblies as fundamental,[7] and considers a voting rule f to be a mapping whose domain is assembly profiles. Nonetheless, as we noted earlier, typical voting procedures aggregate individuals' votes on a seat-by-seat basis, and it is not always clear how to derive a ranking of individual candidates from an assembly ranking. In particular, a voter who perceives strong complementarities among candidates may be unsure how to rank them as individuals.[8] Still, casual observation suggests that voters often have little difficulty in ranking candidates for a given seat independently of the other seats, which suggests that their preferences may be *separable*, as in Definition 1 below.

For $C_i \in \mathbb{C}_i$, $\mathcal{A} = (A_1, ..., A_s) \in \mathbb{A}$, let $(C_i, \mathcal{A}_{-i}) = (A_1, ...A_{i-1}, C_i, A_{i+1}, ..., A_s)$.

Definition 1 *The assembly preferences $L \in \mathbb{L}$ are* ***separable*** *if for all $1 \leq i \leq s$, all $C_i, D_i \in \mathbb{C}_i$, and all $\mathcal{A}, \mathcal{B} \in \mathbb{A}$, $(C_i, \mathcal{A}_{-i}) \succ_L (D_i, \mathcal{A}_{-i})$ implies $(C_i, \mathcal{B}_{-i}) \succ_L (D_i, \mathcal{B}_{-i})$.*

When preferences are separable, an individual who prefers to complete a given assembly with candidate C_i than with candidate D_i, prefers to complete any assembly with C_i.[9] In an obvious sense, we can then say that the

[7] An analogy can be made to consumer theory, where consumers' fundamental preferences are taken to be over consumption bundles, not individual goods.

[8] Austen-Smith and Banks [1991] note that voters with preferences over assemblies may not have well-defined preferences over candidates. Austen-Smith and Banks (1988), analyzes the behavior of voters in a proportional representation system where citizens vote with sophistication on the basis of their predictions about which assembly will be elected.

[9] A more stringent condition is that preferences be *fully separable*: If a *group* of candidates is preferred to another group to complete a particular assembly, then this group is always preferred. All our results and proofs go through unmodified with this stronger notion.

individual prefers candidate C_i to D_i. Formally, let $\mathbb{L}_{sep} \subset \mathbb{L}$ denote the set of separable linear assembly orderings. A separable assembly ranking $L \in \mathbb{L}_{sep}$ *generates* a unique set of candidate rankings R_i, $i = 1, ..., s$ as follows: for $C_i, D_i \in \mathbb{C}_i$, $C_i \succ_{R_i} D_i$ if and only if $(C_i, \mathcal{A}_{-i}) \succ_L (D_i, \mathcal{A}_{-i})$ for some $\mathcal{A} \in \mathbb{A}$.

When assembly preferences are separable, each voter has well-defined preferences over candidates for each seat. Let $\mathbb{R}_i$ denote the linear orderings over $\mathbb{C}_i$, and let $\mathbb{R}_i^n = \mathbb{R}_i \times \cdots \times \mathbb{R}_i$ (n times). An element $R_i \in \mathbb{R}_i^n$ is a profile of candidate orderings for seat i, and an element $R \in \mathbb{R}^{n,s} = \mathbb{R}_1^n \times \cdots \times \mathbb{R}_s^n$ is a profile for each seat. Let $\mathbb{L}_{sep}^n = \mathbb{L}_{sep} \times \cdots \times \mathbb{L}_{sep}$ (n times). An element $L \in \mathbb{L}_{sep}^n$ is a profile of separable assembly orderings. For an assembly profile $L \in \mathbb{L}_{sep}^n$, let $R(L) \in \mathbb{R}^{n,s}$ denote the profiles of candidate orderings for each seat generated by the profile of assembly rankings L. Thus $R_i(L)$ is the profile of candidate orderings generated for seat i, and component $R_{ij}(L)$ is voter j's ranking of the candidates for seat i as generated by his assembly ranking L_j.

While a separable assembly ranking generates a unique candidate ranking, the converse is not true; a single candidate ranking can be generated by many different assembly rankings. For instance the two separable assembly rankings:

Example 2.

I	II
(A_1A_2)	(A_1A_2)
(A_1B_2)	(B_1A_2)
(B_1A_2)	(A_1B_2)
(B_1B_2)	(B_1B_2)

both generate the candidate rankings:

Seat 1	Seat 2
A_1	A_2
B_1	B_2

As we will see, this indeterminacy has important consequences. We say that an assembly ranking is *consistent* with a candidate ranking which it generates.

When preferences are not separable, candidates exhibit interdependencies across seats, and voting on a seat-by-seat basis is obviously problematic. We avoid this immediate problem and focus throughout this paper on the separable case. This restriction only strengthens our results; clearly, if a

seat-based procedure is not efficient or neutral when the domain of preference profiles is restricted to separable preferences, neither will it be so when the domain is unrestricted.

We now define assembly-based and seat-based procedures.

Definition 2 *An **assembly-based** voting rule is a function* $f : \mathbb{L}_{sep}^n \to \mathbb{A}$.

Definition 3 *A **seat-based** voting rule is a function* $f = (f_1, ..., f_s) : \mathbb{R}^{n,s} \to \mathbb{A}$, *where each* f_i *is a function* $f_i : \mathbb{R}_i^n \to \mathbb{C}_i$.

A seat-based voting rule selects a candidate for each seat i based (only) on the voters' rankings of the candidates for that seat.[10] Seat-based voting rules are the rules commonly found in practice. Clearly, a seat-based voting rule is a special case of an assembly-based voting rule, as the following alternative definition makes clear.[11]

Definition 4 *A **seat-based** voting rule is a function* $f : \mathbb{L}_{sep}^n \to \mathbb{A}$, *where* $f(L) = (f_1(R_1(L)), \ldots, f_s(R_s(L)))$.

On the other hand, not every assembly-based rule can be written as a seat-based rule, since the assembly rankings contain more information than the candidate rankings (as demonstrated by Example 2 above).

For $R_i = (R_{i1}, \ldots, R_{in}) \in \mathbb{R}_i^n$, let $H_{R_{ij}}$ denote j's highest ranked candidate for seat i according to R_{ij}.

Definition 5 *Let* $f = (f_1, ..., f_s)$ *be a seat-based voting rule.* f_i *is a **dictatorship for player** j if for every* $R_i \in \mathbb{R}_i^n$, $f_i(R_i) = H_{R_{ij}}$. f *is a **dictatorship** if there exists a voter* $j \in N$, *such that each* f_i *is a dictatorship for* j.

Definition 6 *The assembly-based rule* f *is **efficient** if for every* $L \in \mathbb{L}_{sep}^n$, $f(L)$ *is Pareto optimal.*

Definition 7 *The seat-based rule* f *is **efficient** if for every* $L \in \mathbb{L}_{sep}^n$, $f(R(L))$ *is Pareto optimal.*

[10] We have defined a voting rule to choose exactly one candidate per seat. Allowing for several candidates ("ties") would not affect our results (see also footnote 13).

[11] A seat-based voting rule is a special case of an assembly-based voting rule even when preferences are not separable, provided that one specifies a single-valued mapping from assembly rankings to candidate rankings.

Implicit in these definitions is the presumption that individuals vote non-strategically with respect to their assembly and candidate preferences. That is, voters rank the assemblies according to their true assembly rankings, and rank the candidates according to their generated candidate rankings.[12] Allowing for strategic voting would not aid in resolving the issues we discuss.[13]

2 Designated-Seat Assemblies

2.1 Efficiency

Consider a two-seat election with two candidates per seat, and an odd number of voters greater than two. All voters have separable preferences. Suppose that both seats are decided by plurality elections. It is easy to see that at least one voter must have her first choice elected in each seat, and thus must have her favorite assembly chosen. The election is therefore efficient.[14] This situation is rather limited in scope, however. The following theorem shows that with more seats, or more candidates, the only efficient voting method is a dictatorship.

Theorem 1 *Let the domain of preferences be $\mathbb{L}_{sep}$. Consider a designated-seat election with at least one voter and at least two candidates per seat. Suppose there are*
a) at least three seats, or
b) at least two seats, and at least three candidates for some seat.
Then, the only efficient seat-based voting rule is a dictatorship.

2.2 Neutrality

Neutrality requires that if outcome A is chosen at profile P, and P' is obtained from P by permuting A and B in everyone's ranking, then B be chosen at Profile P'. This (standard) statement makes no reference to whether A is an

[12] With respect to the assembly preferences, this non-strategic voting is sincere voting. With respect to the candidate preferences, this voting is a natural extension of sincere voting (see Benoît and Kornhauser (1991) and (1995) for a fuller discussion of this type of candidate voting, where it is termed *simple*.)

[13] See Section 4.1 for a fully game-theoretic model.

[14] Oskal-Sanver and Sanver (2006) consider further properties of the two-candidate, two-seat case.

individual candidate or an assembly. Nevertheless, a difficulty arises in the case of assemblies: permuting assemblies in voters' separable rankings may not be consistent with maintaining the separability of these rankings.

Consider an election for a two-seat assembly, with two candidates per seat, and two voters with the separable assembly rankings:

Example 3.

Voter 1	Voter 2
(A_1A_2)	(B_1B_2)
(A_1B_2)	(B_1A_2)
(B_1A_2)	(A_1B_2)
(B_1B_2)	(A_1A_2)

Profile I

and corresponding seat rankings:

Voter 1		Voter 2	
Seat 1	Seat 2	Seat 1	Seat 2
A_1	A_2	B_1	B_2
B_1	B_2	A_1	A_2

The assemblies (A_1B_2) and (A_1A_2) cannot be swapped, ceteris paribus, in the voters' rankings without violating the separability of the preferences. A straightforward resolution of this problem is to consider only those permutations which preserve the separability of the voters' preferences, as in the following definition:

Definition 8 *For any $L \in \mathbb{L}_{sep}^n$, let $\sigma(L) = (\sigma(L_1), \ldots, \sigma(L_n))$, where $\sigma : \mathbb{A}_s \to \mathbb{A}_s$ is a permutation of the assemblies. The assembly-based voting rule f is* ***s-neutral*** *if for all $L \in (\mathbb{L}_{sep})^n$, $f(\sigma(L)) = \sigma^{-1}(f(L))$ whenever $\sigma(L) \in \mathbb{L}_{sep}^n$. The seat-based voting rule f is s-neutral if for all $L \in (\mathbb{L}_{sep})^n$, $f(R(\sigma(L))) = \sigma^{-1}(f(R(L)))$ whenever $\sigma(L) \in \mathbb{L}_{sep}^n$.*

In the case of Profile I, s-neutrality allows us to consider, among other things, a swap of (A_1A_2) for (B_1B_2), and a swap of (A_1B_2) for (B_1A_2), both of which preserve the separability of the voters' preferences. The property s-neutrality requires that if (A_1A_2) is selected with Profile I above, then (B_1B_2) be chosen with the profile I':

Voter 1	Voter 2
(B_1B_2)	(A_1A_2)
(A_1B_2)	(B_1A_2)
(B_1A_2)	(A_1B_2)
(A_1A_2)	(B_1B_2)

Profile I'

Similarly, s-neutrality requires that if (A_1B_2) is chosen with profile I, then (B_1A_2) be chosen with the profile I":

Voter 1	Voter 2
(A_1A_2)	(B_1B_2)
(B_1A_2)	(A_1B_2)
(A_1B_2)	(B_1A_2)
(B_1B_2)	(A_1A_2)

Profile I"

No seat-based voting system can accomplish this second transformation, since assembly profiles I and I" yield the same seat profiles. Therefore, no s-neutral seat-based rule can select (A_1B_2) with Profile I. On the other hand, the first transformation can be accomplished, and (A_1A_2) can be chosen by an s-neutral rule. For instance, the seat-based anti-dictatorship that always selects Voter 2's bottom candidate for each seat is s-neutral, and selects (A_1A_2) with Profile I, and (B_1B_2) with Profile I'. Of course, an anti-dictatorship is not encountered in practice. As Theorem 2 indicates, there is a good reason we have had recourse to a theoretical rule.

Note that, although Theorem 1 shows that no existing voting rule is efficient with respect to assemblies, typical voting rules are efficient *seat-by-seat*. That is, typical voting rules will exclude a candidate A from an assembly if all voters rank a candidate B above it. Formally:

Definition 9 *A voting rule $f = (f_1, ..., f_s)$ is* ***seat-by-seat efficient*** *if for all $R = (R_1, ..., R_s) \in \mathbb{R}^{n,s}$, and all $i = 1, ..., s$, $A_i \succ_{R_{ij}} B_i$ for all $j = 1, ..., n$, implies that $f_i(R_i) \neq B_i$.*

The following theorem shows that no seat-by-seat efficient voting rule, other than a dictatorship, is s-neutral.[15,16]

Theorem 2 *Let the domain of preferences be $\mathbb{L}_{sep}$. Consider a designated-seat election with at least one voter, at least three seats, and at least two*

[15] In single-candidate elections, it may be difficult to obtain neutrality, efficiency, and anonymity for social choice *functions* (see Moulin (1983)). Note, however, that we have not imposed anonymity here. More importantly, Theorem 2 remains true *exactly as stated* if we allow for correspondences (although the analysis is then more involved). This is because the non-neutrality is not driven by difficulties involving ties. We note that Theorem 1 is also unchanged if we allow for correspondences.

[16] Although it seems that Theorem 2 should extend to two seats, as in Theorem 1, we have been unable to establish this.

candidates per seat. The only s-neutral, seat-by-seat efficient, seat-based voting rule is a dictatorship.

Although efficiency and neutrality are, on the face of it, unrelated concepts Theorems 1 and 2 are closely connected; both stem from the fact that several assembly rankings are consistent with a single set of seat rankings, and their proofs in the appendix are virtually identical.

When the outcomes are individual candidates, the appeal of neutrality is obvious. After all, swapping candidates A and B in the voters' rankings amounts to a mere relabeling of the alternatives.[17] The situation is more subtle in the case of assemblies. When assembly (A_1A_2) is swapped with (B_1B_2) in the voters' rankings, holding the other assemblies fixed, it is difficult to interpret this as a mere relabeling of the assemblies, since the component candidates have not been relabeled. A pure relabeling would, say, relabel A_1 as B_1, and A_2 as B_2, so that (A_1B_2) and (B_1A_2) would also have to be swapped in the voters' rankings, along with (A_1A_2) and (B_1B_2). This relabeling point of view suggests a definition of neutrality in which the permutations of assemblies is further restricted to only those that can be accomplished through the permutation of the candidates. Any voting rule that is neutral on a seat-by-seat basis, will be assembly neutral in this more restricted sense, and so this type of neutrality can be obtained. However, our definition of s-neutrality seems, to us, more in keeping with the standard Social Choice Theory approach, which is outcome-based and emphasizes the ordinality of preference rankings, while allowing for domain restrictions (e.g., single-peakedness). The reader can judge the two notions by reconsidering profiles I and I'. Our notion of s-neutrality requires that if (A_1A_2) is chosen with profile I, then (B_1B_2) be chosen with profile I', while the more restrictive notion just outlined would impose no requirement. We believe that the change from (A_1A_2) to (B_1B_2) is called for in a "neutral" rule, since the ordinal information about (A_1A_2) in profile I corresponds to the ordinal information about (B_1B_2) in profile I'.

[17] Of course, there are some situations where neutrality may not be desired, such as when status quo status is deemed important.

3 At-Large Assemblies

We now briefly turn our attention to at-large assemblies, where similar difficulties arise.

In an at-large election, candidates do not declare for a particular seat. If $\mathbb{C}$ is the set of candidates, then an s-sized assembly is any subset of $\mathbb{C}$ of cardinality s. Let $\mathcal{A}_{ij}$ and $\mathcal{B}_{ij}$ be two (sub) assemblies of size $s-1$, neither of which contain candidate A_i or A_j. Preferences are *separable* if $\{A_i\} \cup \mathcal{A}_{ij} \succ \{A_j\} \cup \mathcal{A}_{ij}$ implies $\{A_i\} \cup \mathcal{B}_{ij} \succ \{A_j\} \cup \mathcal{B}_{ij}$. Again, separability leads to a well-defined ranking of the candidates, but several assembly rankings are consistent with a given candidate ranking (see Benoît and Kornhauser (1991, 1999) for more details).

In a *candidate-based* procedure, each voter submits a ranking of the candidates. Suppose that there are six candidates vying for a position on a three-seat assembly, and that all voters have separable assembly preferences. The voters divide into three equally-sized groups with the following generated candidate preferences:

Example 4.

Group I	Group II	Group III
A	B	C
B	C	A
D	F	E
E	D	F
F	E	D
C	A	B

Suppose, as is common, that a plurality over candidates is used, with each voter being given either one, two, or three votes to cast for different candidates. In all three cases, the assembly (ABC) is easily elected. (ABC) is also elected using a Borda Count over candidates, single transferable voting, or any Condorcet consistent method. Nevertheless, all voters may prefer (DEF) to (ABC) (for instance, every voter may have an intense dislike for his or her least favorite candidate, but view the other candidates about equally).

More generally, consider any voting rule that selects at least three candidates from the above candidate rankings.[18] Call the rule inefficient if a

[18] If the rule selects three candidates, they form the assembly. If the rule selects more than three, the assembly will (somehow) be formed from the selected candidates.

Pareto inferior assembly can be formed from the selected listed. Suppose the rule is anonymous and neutral with respect to the individual candidates. If the system selects any one of A, B and C, then it must select all three. If the system selects any one of D, E and F, then, again, it must select all three. Either selection may be inefficient, and leads to a non-neutrality with respect to the assemblies.

This example points to an analogue of Theorems 1 and 2, at least for an important class of candidate-based procedures. Indeed, in Benoît and Kornhauser (1994) we show that all constant scoring systems are inefficient and non-neutral.[19] However, although every at-large candidate-based voting system we know of is inefficient, we have been unable to establish results for at-large assemblies of the generality of Theorems 1 and 2.

To appreciate the nature of the difficulty, let us reconsider our analysis of designated-seat assemblies. In an assembly-based procedure voters rank the assemblies, whereas in a seat-by-seat procedure voters rank the candidates for individual seats. There is another less obvious, but also important distinction: With seat-by-seat procedures, the seats are decided independently of each other.[20] To see the role played by this feature, consider a rule which (i) asks voters to rank the candidates for each seat, then (ii) for each voter, looks at the group of candidates the voter has ranked first,[21] and finally (iii) selects as an assembly that group which is ranked first most often (with a tie-breaking rule if necessary). It is easy to see that while this rule only asks for candidate information, it is equivalent to a plurality rule in which voters are asked to rank their assemblies. Therefore, this rule is efficient. It is also essentially an assembly-based rule in disguise. The different seats in a designated-seat allow us to exclude "disguised assembly rules", but it is unclear (at least to us) how to rule out such rules in the case of at-large assemblies.

[19] A constant scoring system is one in which voters get k votes to cast for k different candidates. Theorem 1 in Benoît and Kornhauser (1994) shows inefficiency. Although a non-neutrality result is not stated, the proof of Theorem 1 also establishes the non-neutrality of constant scoring systems.

[20] Oskal-Sanver and Sanver (2006) makes a similar observation.

[21] For instance, in Example 1 this group would be (B_S, A_J, A_F) for Voter 1.

4 Discussion

Our conclusion that seat-based procedures are neither efficient nor neutral raises several questions. Here, and in the next subsection, we consider four of them: How pathological can seat-based electoral results be? What further restrictions on assembly preferences will guarantee efficiency? Could endogenizing the set of candidates guarantee efficiency? Would it help if voters behaved strategically?

We first show that seat-based procedures may yield quite perverse results. Consider a designated-seat election with two seats and three candidates per seat, in which each seat is decided by a plurality election. To begin, let us examine a non-separable case, which has been hitherto excluded. Suppose that Voter 1 has the following non-separable preferences:

	VOTER 1
$1st$	A_1A_2
$2nd$	B_1A_2
$3rd$	C_1A_2
$4th$	B_1C_2
$5th$	C_1C_2
$6th$	B_1B_2
$7th$	C_1B_2
$8th$	A_1C_2
$9th$	A_1B_2

Candidate A_1 is both on Voter 1's favorite assembly and least favorite assembly, so that it is unclear how he should vote, even if he is just trying to vote "sincerely". It is immediately obvious that seat-by-seat voting may not be a good idea if many voters have non-separable preferences like these. Indeed, suppose that the population divides into four equally-sized groups with the following partially listed preferences:

	Group 1	**Group 2**	**Group 3**	**Group 4**
$1st$	A_1A_2	A_1C_2	C_1B_2	B_1B_2
$2nd$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$9th$	A_1B_2	A_1B_2	A_1B_2	A_1B_2

Let us assume that individuals vote for their favorite assembly, seat by seat. Groups 1 and 2 both vote for A_1 for seat 1, while groups 3 and 4 both vote for B_2 for seat 2. Every other candidate receives votes from at most one group. The winning assembly is A_1B_2 even though it is bottom-ranked by every voter!

If preferences are separable, such an extreme pathology is not possible, since an individual who votes for a winning candidate cannot rank the winning assembly last. Nonetheless, as we have already seen, the resulting assembly may be inefficient. While this is in and of itself a bad thing, the reader may still wonder just how poor the result can be. The next example shows that the problem may be quite severe.

Groups 1, 3, and 4 are equally-sized, while Group 2 is larger by one. All voters have separable preferences. We list these below, along with the generated candidate rankings.

Example 5.

Assembly Preferences

	Group 1	**Group 2**	**Group 3**	**Group 4**
$1st$	A_1A_2	A_1C_2	C_1B_2	B_1B_2
$2nd$	A_1C_2	C_1C_2	C_1C_2	B_1C_2
$3rd$	C_1A_2	B_1C_2	C_1A_2	C_1B_2
$4th$	C_1C_2	A_1A_2	B_1B_2	C_1C_2
$5th$	B_1A_2	B_1A_2	B_1C_2	B_1A_2
$6th$	B_1C_2	C_1A_2	B_1A_2	B_1C_2
$7th$	A_1B_2	A_1B_2	A_1B_2	A_1B_2
$8th$	C_1B_2	C_1B_2	A_1C_2	A_1C_2
$9th$	B_1B_2	B_1B_2	A_1A_2	A_1A_2

Candidate Preferences

	Group 1		**Group 2**		**Group 3**		**Group 4**	
	Seat 1	Seat 2	Seat 1	Seat 2	Seat 1	Seat 2	Seat 1	Seat 2
$1st$	A_1	A_2	A_1	C_2	C_1	B_2	B_1	B_2
$2nd$	C_1	C_2	C_1	A_2	B_1	C_2	C_1	C_2
$3rd$	B_1	B_2	B_1	B_2	A_1	A_2	A_1	A_2

A seat-by-seat plurality results in A_1B_2. However, A_1B_2 is only ranked seventh out of ten by every voter. In contrast, the assembly C_1C_2 is ranked second by about half the voters and no lower than fourth, while the assembly A_1C_2 is a Condorcet winner among assemblies. Note that C_1C_2 would result

from a Borda count over assemblies, while a plurality election over assemblies would yield A_1C_2.

Although the assumption of separable assembly preferences guarantees that voters have well-defined candidate preferences, and, in this sense, rationalizes seat-by-seat voting, it is not sufficient to guarantee that seat-by-voting is desirable.[22] Perforce, neither is a weaker assumption. We next consider a stronger restriction.

In many elections, it is plausible to suppose that voters assign a common order of importance to the various seats. For instance, they may all agree that the mayor is more important than the district attorney, who in turn is more important than the police chief. Suppose further that voters behave lexicographically with respect to this order, as in the following definition:

Definition 10 *A voter's preferences are said to be* ***top-lexicographic*** *if there is a seat order* $(1, ..., s)$ *such that the voter always prefers assembly* $\mathcal{A} = (A_1, ..., A_S)$ *to assembly* $\mathcal{A}$*' whenever* $\mathcal{A}$ *and* $\mathcal{A}$*' first differ in seat* j*, and* A_j *is the voter's top-ranked candidate for seat* j*.*

Benoît and Kornhauser (1994, theorem 5) shows that when all voters have top-lexicographic preferences with respect to a common seat order, seat-by-seat plurality rule always selects an efficient assembly. However, although top-lexicographicity has a certain appeal, it is a very strong assumption. Note that even if there is a clear sense in which one seat is much more important than another, a voter's preferences will still likely not be top-lexicographic if she is almost indifferent between her top two candidates for some seat.[23]

4.1 A Game-Theoretic Model

Up to now, the candidates have been exogenously given, and the voters have behaved sincerely. In this section, we show that these features are not the source of our difficulties. Specifically, we show that in an election game in which each candidate strategically adopts a position and each voter votes strategically, the result may still be inefficient.

[22] Though the assumption of separable assembly preferences is a strong one (see, for instance Benoît and Kornhauser (1991, 1999) and, in a different electoral context, Brams et al. (1997)), it is a reasonable one in many situations.

[23] Another restriction considered in Benoît and Kornhauser (1994) is *1-blockness*. This restriction is also strong.

A typical problem in voting games is a surfeit of equilibria. To circumvent this problem we now assume that there are two candidates per seat (but they may adopt many positions) and that the voting rule is monotonic (defined below). This enables a unique subgame perfect equilibrium outcome in undominated strategies. Relaxing these assumptions typically results in a multiplicity of trivial equilibria, many of which are inefficient.

Formally, suppose there are $s \geq 3$ seats and two candidates per seat. Let $\mathbb{C}_1, ..., \mathbb{C}_s$ be a collection of finite sets, such that for each set $|\mathbb{C}_i| \geq 2$. We interpret $\mathbb{C}_i$ as the set of positions that a candidate for seat i can adopt, and we will identify each candidate with the position that she adopts. Thus, we interpret $\mathbb{A} = \mathbb{C}_1 \times \cdots \times \mathbb{C}_s$ as the set of possible assemblies. Let N be the set of voters. We assume that voters have separable rankings over the assemblies. Since there are only two candidates per seat, for each seat every voter is called upon to rank only the two positions that present themselves. Accordingly, each decision rule f_i takes as its domain the profiles of rankings of any two positions of seat i (rather than the rankings of all the positions of seat i). The timing of the *positional voting game* is as follows.

1. First, for each seat i, both candidates for that seat choose an element of $\mathbb{C}_i$.

2. Second, for each seat i, every voter submits a ranking of the positions the candidates have chosen for that seat.

3. Third, the seat-by-seat rule $f = (f_1, ..., f_s)$ selects a candidate for each seat based on these rankings. If there are two candidates at the same position, the rule chooses one of them with a 50% chance, otherwise the rule is deterministic.

We now define *monotonicity*, a common property of voting rules, especially when there are only two candidates per seat.[24] As discussed above, the only role monotonicity – coupled with the assumption of two candidates per seat – plays here is to yield uniqueness in undominated strategies.

Definition 11 *The rule $f = (f_1, ..., f_s)$ is* ***monotonic*** *if for all $A_i \in \mathbb{C}_i$, for all $R_i \in \mathbb{R}_i$, we have $f_i(R_i) = A_i \Rightarrow f_i(R_i') = A_i$ whenever R_i' is derived from R_i by raising A_i in some rankings R_{ij}, ceteris paribus.*

[24] With two candidates, majority rule is monotonic, as well as variants which weight voters differently, or favour certain candidates.

A non-trivial voting rule satisfies voter sovereignty:

Definition 12 *The rule $f = (f_1, ..., f_s)$ satisfies* ***voter sovereignty*** *if for every f_i, and any $A_i, B_i \in \mathbb{C}_i$, there exists a set of voter rankings of A_i and B_i such that A_i is elected, and a set of voter rankings such that B_i is elected.*

Voter sovereignty is, of course, an extremely weak assumption. Without this assumption we would still obtain a (modified) inefficiency result.

Let $\{\mathbb{C}_1, ..., \mathbb{C}_s, N, f\}$ be the *positional voting game form* associated with the above defined positional voting game. The game form represents the game before the voters' preferences have been specified. The following definition provides a fairly strong notion of inefficiency.

Definition 13 *The positional voting game form $\{\mathbb{C}_1, ..., \mathbb{C}_s, N, f\}$ is* ***inefficient*** *if there exists a separable preference profile for which, in every subgame perfect equilibrium in undominated strategies of the associated positional voting game, a Pareto inferior assembly is elected.*

Proposition 1 *Let $\{\mathbb{C}_1, ..., \mathbb{C}_s, N, f\}$ be a positional voting game form, with $s \geq 3$, $|\mathbb{C}_i| \geq 2$, $i = 1, ..., s$. Suppose that f satisfies monotonicity and voter sovereignty, and that f is not a dictatorship. Then $\{\mathbb{C}_1, ..., \mathbb{C}_s, N, f\}$ is inefficient.*

As an application of Proposition 1, consider a positional voting game in which there are three seats and three voters. For each seat i, $\{0, 1\} \subset \mathbb{C}_i \subset \mathbb{R}_i$. Every voter has single-peaked preferences with respect to each seat. Voter 1's ideal positions are $(0, 1, 1)$; Voter 2's ideal positions are $(1, 0, 1)$; Voter 3's ideal positions are $(1, 1, 0)$. Each seat is decided by majority rule. There are two candidates per seat, each of whom picks a position in the first stage of the game, after which every voter casts a vote for each seat. It is easily verified that in the unique undominated subgame perfect equilibrium, each candidate chooses the position 1. The resultant assembly is $\{1, 1, 1\}$, although it may well be that every voter prefers the assembly $\{0, 0, 0\}$ to the assembly $\{1, 1, 1\}$.[25]

Note that the voters' preferences in this application are well-behaved in that each voter has single-peaked preferences over each candidate. On the

[25] As this example shows, the assumption that each $\mathbb{C}_i$ is finite is not critical. A very similar example appears in Benoît and Kornhauser (1994).

other hand, the voter's preferences over assemblies are not single-peaked. More to the point, there is no assembly that is a Condorcet winner. This is not surprising, given the multi-dimensional nature of the assemblies.[26] Although it is quite strong to assume the existence of a Condorcet assembly, it is instructive to consider the implications of this assumption. If there is a Condorcet assembly, then each member of that assembly is a Condorcet winner for her seat. That is, if A_i is a member of a Condorcet assembly, then A_i is preferred by a majority of voters to every other candidate for seat i. (On the other hand, as the previous example shows, even if each seat has a Condorcet winning candidate, the resulting assembly need not be a Condorcet winning assembly). Therefore, if a Condorcet assembly always exists, then any rule $f = (f_1, ..., f_s)$ where each f_i is Condorcet consistent is efficient. While this is a positive result, we note two caveats. The first, which has already been noted, is that positing the existence of a Condorcet winner, which is always a strong assumption, is especially strong here. The second is that most voting rules are not seat-by-seat Condorcet consistent. Note that in Example 5, although there is a Condorcet assembly, and hence a Condorcet set of candidates, these candidates are not chosen by the seat-by-seat plurality rule used.

5 Conclusion

Strictly speaking, selecting an efficient outcome is not likely to be a problem in an election with a large population, even for the most absurd voting system. The reason is simply that with thousands, or millions, of heterogeneous voters, almost inevitably every outcome will be someone's favorite assembly.[27] Nevertheless, Theorem 1 casts doubt on common electoral procedures: If efficiency cannot be guaranteed, there seems to be little reason to believe that the elected assembly will be desirable. Or, if there is such a reason, it remains to be articulated.

At an abstract level, our results emphasize that it is misleading to analyze

[26] Indeed, it is well-known that even if voters' preferences over $\mathbb{R}^n$, $n \geq 2$, are single-peaked in each dimension, there will generally not be a Condorcet winner.

[27] At least, every outcome will inevitably be some voter's favourite when there is a relatively small number of assemblies. This reasoning does not apply for the U.S. House of Representatives which has 435 seats and 2^{435} possible different assemblies (with a strict two party system).

a seat-by-seat election in terms of the properties of the voting rules of the individual seats.

6 Appendix

In order to prove Theorems 1 and 2, we first establish a lemma.

For $1 \leq i \leq s$, we say that $P_i : \mathbb{C}_i \to \mathbb{R}$ is a **candidate point assignment** if for all $A_i, B_i \in \mathbb{C}_i$, $A_i \neq B_i \Rightarrow P_i(A_i) \neq P_i(B_i)$. A set of candidate point assignments yields candidate rankings and assembly rankings as per the following definition:

Definition 14 *Let $P_i, ..., P_s$ be point assignments. We say that P_i* ***yields*** *the (strict) candidate ranking R_i if for any $A_i, B_i \in \mathbb{C}_i$, $A_i \succ_{R_i} B_i$ if $P_i(A_i) > P_i(B_i)$. We say that $P_i, ..., P_s$* ***yields*** *the (strict) assembly ranking L if for any $\mathcal{A} = (A_1, ..., A_s) \in \mathbb{A}, \mathcal{B} = (B_1, ..., B_s) \in \mathbb{A}$, $\mathcal{A} \succ_L \mathcal{B}$ if $\sum_{i=1}^{s} P_i(A_i) > \sum_{i=1}^{s} P_i(B_i)$.*

The following lemma shows that a set of candidate point assignments yields a separable assembly ranking.

Lemma 1 *If the point assignments $P_1, ..., P_s$ yield the strict assembly ranking L, then L is separable.*

Proof. *Obvious* ■

Proof of part a) of Theorem 1 and Theorem 2. For ease of exposition, we analyze the case of 3 candidates per seat. The modifications needed for an arbitrary number of candidates are trivial.[28] Obviously, any efficient rule must be seat-by-seat efficient. Therefore, it is sufficient to prove that any non-dictatorial, seat-by-seat efficient rule $f = (f_1, \ldots, f_s)$ is neither efficient nor s-neutral.

Proof of a). Suppose that $f = (f_1, \ldots, f_s)$ is a non-dictatorial, seat-by-seat efficient rule.

[28] In particular, in the subsequent profiles any additional candidates would be ranked below the three candidates A_i, B_i, C_i. Any seat with only two candidates would have the bottom-ranked candidate deleted from the profiles.

For seat i, $1 \leq i \leq s$, consider the candidate preference profile $R_i^0 \in \mathbb{R}_i^n$, defined by$R_i^0 =$

Voter 1	Voter 2		Voter n
A_i	A_i	$\cdots$	A_i
B_i	B_i	$\cdots$	B_i
C_i	C_i	$\cdots$	C_i

From seat-by-seat efficiency, $f_i\left(R_i^0\right) = A_i$. For $1 \leq j \leq n$, let the profile R_i^j be obtained from R_i^0 by raising B_i, ceteris paribus, in the rankings of voters $1, .., j$. Thus, for instance, $R_i^2 =$

Voter 1	Voter 2	Voter 3		Voter n
B_i	B_i	A_i	$\cdots$	A_i
A_i	A_i	B_i	$\cdots$	B_i
C_i	C_i	C_i	$\cdots$	C_i

Because of seat-by-seat efficiency, $f_i\left(R_i^j\right) = A_i$ or B_i for $1 \leq j < n$, and $f_i\left(R_i^n\right) = B_i$. Let voter $1 \leq k_i \leq n$ be such that $f_i\left(R_i^j\right) = A_i$ for $j = 0, ..., k_i - 1$,while $f_i\left(R_i^{k_i}\right) = B_i$.

First suppose that there exist $1 \leq i, j \leq s$, such that $k_i < k_j$, and suppose w.l.o.g. that $i = 1, j = 2$. We have

$$\begin{aligned} & f\left(R_1^{k_1}, R_2^{k_1}, R_3^0, \cdots, R_s^0\right) \\ = \; & \left(f_1\left(R_1^{k_1}\right), f_2\left(R_2^{k_1}\right), f_3\left(R_3^0\right), \ldots, f_s\left(R_s^0\right)\right) \\ = \; & \left(B_1, A_2, A_3, \ldots, A_s\right) \end{aligned}$$

We now use point assignments to find two sets of separable assembly rankings consistent with the candidate rankings $\left(R_1^{k_1}, R_2^{k_1}, R_3^0, \cdots, R_s^0\right)$. Firstly, the point assignments :

For Voter j=1,...,k_1

	Seat 1	Seat 2	Seats $i = 3, ..., s$
Points	$B_1 : 10$	$B_2 : 10$	$A_i : 50$
Points	$A_1 : 5$	$A_2 : 4$	$B_i : 1$
Points	$C_1 : 0$	$C_2 : 0$	$C_i : 0$

For Voter j=k_1+1,...,n

	Seat 1	Seat 2	Seats $i = 3, ..., s$
Points	$A_1 : 10$	$A_2 : 10$	$A_i : 50$
Points	$B_1 : 4$	$B_2 : 5$	$B_i : 1$
Points	$C_1 : 0$	$C_2 : 0$	$C_i : 0$

yield the candidate rankings $\left(R_1^{k_1}, R_2^{k_1}, R_3^0, \cdots, R_s^0\right)$ and the (partially listed)

assembly rankings:

$$\begin{array}{cc} \text{Voter } j=1,...,k_1 & \text{Voter } j=k_1+1,...,n \\ (B_1B_2A_3\cdots A_s) & (A_1A_2A_3\cdots A_s) \\ (A_1B_2A_3\cdots A_s) & (A_1B_2A_3\cdots A_s) \\ (B_1A_2A_3\cdots A_s) & (B_1A_2A_3\cdots A_s) \\ \vdots & \vdots \end{array}$$

Since $f\left(R_1^{k_1}, R_2^{k_1}, R_3^0, \cdots, R_s^0\right) = (B_1A_2A_3\cdots A_s)$ although everyone prefers $(A_1B_2A_3\cdots A_s)$ to $(B_1A_2A_3\cdots A_s)$, f is inefficient.

Secondly, the point assignments

For Voter j=1,...,k_1

	Seat 1	Seat 2	Seats $i=3,...,s$
Points	$B_1:10$	$B_2:10$	$A_i:50$
Points	$A_1:4$	$A_2:5$	$B_i:1$
Points	$C_1:0$	$C_2:0$	$C_i:0$

For Voter j=k_1+1,...,n

	Seat 1	Seat 2	Seats $i=3,...,s$
Points	$A_1:10$	$A_2:10$	$A_i:50$
Points	$B_1:5$	$B_2:4$	$B_i:1$
Points	$C_1:0$	$C_2:0$	$C_i:0$

still yield the candidate rankings $\left(R_1^{k_1}, R_2^{k_1}, R_3^0, \cdots, R_s^0\right)$, but now yield the (partial) assembly rankings:

$$\begin{array}{cc} \text{Voter } j=1,...,k_1 & \text{Voter } j=k_1+1,...,n \\ (B_1B_2A_3\cdots A_s) & (A_1A_2A_3\cdots A_s) \\ (B_1A_2A_3\cdots A_s) & (B_1A_2A_3\cdots A_s) \\ (A_1B_2A_3\cdots A_s) & (A_1B_2A_3\cdots A_s) \\ \vdots & \vdots \end{array}$$

The rule f still chooses $(B_1, A_2, A_3, \ldots, A_s)$, although $(B_1, A_2, A_3, \ldots, A_s)$ and $(A_1, B_2, A_3, \ldots, A_s)$ have been swapped in everybody's assembly ranking. Therefore, f is not s-neutral.

Now suppose that $k_i = k$ for all $1, 2, ..., s$. Since k is not a dictator, there exists an f_i and an $R_i \in \mathbb{R}_i^n$ such that $f_i(R_i) \neq H_{R_{ik}}$. W.l.o.g., let $f_i = f_3$ and let R_3 be such that $f_i(R_3) \neq H_{R_{3k}}$.

For $\left(R_1^{k-1}, R_2^k, R_3, R_4^0, \cdots, R_s^0\right)$ we have

$$\begin{aligned} & f\left(R_1^{k-1}, R_2^k, R_3, R_4^0, \cdots, R_s^0\right) \\ = \; & \left(f_1\left(R_1^{k-1}\right), f_2\left(R_2^k\right), f_3\left(R_3\right), f_4\left(R_4^0\right), \ldots, f_s\left(R_s^0\right)\right) \\ = \; & (A_1, B_2, f_3(R_3), A_4, \ldots, A_s). \end{aligned}$$

Let voter k's candidate and assembly rankings be derived from the point assignment:

For Voter j=k

	Seat 1	Seat 2	Seat 3	Seats $i = 4, ..., s$
Points	$A_1 : 100$	$B_2 : 100$	$H_{R_{3k}} : 200$	$A_i : 500$
Points	$B_1 : 50$	$A_2 : 50$	$f_3(R_3) : 100 + \varepsilon$ $X_3 : M$	$B_i : 1$
Points	$C_1 : 0$	$C_2 : 0$		$C_i : 0$

where $X_3 \in \{A_3, B_3, C_3\}$, $X_3 \neq H_{R_{3k}}$ or $f_3(R_3)$, and $M = 150$ if voter k ranks X_3 above $f_3(R_3)$, while $M = 50$ if k ranks X_3 below $f_3(R_3)$. Note that if ε were equal to 0, then this putative point assignment would yield assembly rankings in which $(A_1, B_2, f_3(R_3), A_4, \ldots, A_s)$ and $(B_1, A_2, H_{R_{3k}}, A_4, \ldots, A_s)$ were tied in the voter's ranking. Choosing ε_k slightly above 0 or slightly below 0, flips these two assemblies in the voters' rankings, without changing any other assembly rankings and without changing the candidate rankings.

Now partition voters 1,..., $k-1$, into the two sets V_I and V_{II} defined by $j \in V_I$ if j ranks $f_3(R_3)$ below $H_{R_{3k}}$, and $j \in V_{II}$ if j ranks $f_3(R_3)$ above $H_{R_{3k}}$.

For a voter $j \in V_I$, let the candidate and assembly rankings be derived from the partially listed point assignment:[29]

Voter j=1,...,k−1

	Seat 1	Seat 2	Seat 3	Seats $i = 4, ..., s$
Points	$B_1 : 100$	$B_2 : 100$		$A_i : 500$
Points	$A_1 : 50$	$A_2 : 45$	$H_{R_{3k}} : 10$ $f_3(R_3) : 5 + \varepsilon$	$B_i : 1$
Points	$C_1 : 0$	$C_2 : 0$		$C_i : 0$

Note that if ε were equal to 0, then this point assignment would yield assembly rankings in which $(A_1, B_2, f_3(R_3), A_4, \ldots, A_s)$ and $(B_1, A_2, H_{R_{3k}}, A_4, \ldots, A_s)$ were tied in the voter's ranking. Choosing ε slightly above 0 or slightly below 0, flips these two assemblies in the voters' ranking, without changing any other assembly rankings and without changing the candidate rankings.

[29] To complete the point assignment, the remaining point(s) must be chosen so that no assemblies are tied, and the candidate ranking is respected. For instance if $H_{R_{3k}} \neq H_{R_{3j}}$, then we could have $H_{R_{3j}} = 100$.

For a voter $j \in V_{II}$, let the candidate and assembly rankings be derived from the partially listed point assignment

	Voter $j=1,...,k-1$			
	Seat 1	Seat 2	Seat 3	Seats $i = 4, ..., s$
Points	$B_1 : 100$	$B_2 : 100$		$A_i = 500$
Points	$A_1 : 45$	$A_2 : 50$	$H_{R_{3k}} : 5$ $f_3(R_3) : 10 + \varepsilon$	$B_i = 1$
Points	$C_1 : 0$	$C_2 : 0$		$C_i = 0$

Again, if ε were equal to 0, then this point assignment would yield assembly rankings in which $(A_1, B_2, f_3(R_3), A_4, \ldots, A_s)$ and $(B_1, A_2, H_{R_{3k}}, A_4, \ldots, A_s)$ were tied in the voter's ranking, while choosing ε slightly above 0 or slightly below 0, flips these two assemblies in the voters' ranking, without changing any other assembly rankings and without changing the candidate rankings.

If $k < n$, proceed in a similar fashion for voters $k+1, ..., n$.[30]

Now, choosing $\varepsilon < 0$ small enough, yields the candidate rankings $\left(R_1^{k-1}, R_2^k, R_3, R_4^0, \cdots, R_s^0\right)$, and assembly rankings in which everyone prefers $(B_1, A_2, H_{R_{3k}}, A_4, \ldots, A_s)$ to $(A_1, B_2, f_3(R_3), A_4, \ldots, A_s)$. Since $f\left(R_1^{k-1}, R_2^k, R_3, R_4^0, \cdots, R_s^0\right) = (A_1, B_2, f_3(R_3), A_4, \ldots, A_s)$, f is inefficient. Choosing $\varepsilon > 0$ small enough yields the same candidate ranking, and hence the same assembly choice, but swaps $(A_1, B_2, f_3(R_3), A_4, \ldots, A_s)$ and $(B_1, A_2, H_{R_{3k}}, A_4, \ldots, A_s)$ in everybody's rankings. Hence f is not s-neutral.

Proof of part b) of Theorem 1. Part a) establishes the theorem for $s > 2$, therefore consider $s = 2$.

Suppose that, say, f_1 is a dictatorship for voter 1. Consider the seat 1 profile $R_1 =$

Voter 1	Voter 2		Voter n
A_1	B_1	$\cdots$	B_1
$\vdots$	$\vdots$	$\cdots$	$\vdots$

We have $f_1(R_1) = A_1$. Since f is not a dictatorship, f_2 is not a dictatorship for voter 1. Let R_2 be a profile for seat 2 such that $f_2(R_2) \neq H_{R_{21}}$. Con-

[30] For instance, one subset of voters will receive the point assignment

	For Voter $j=k+1,...,n$			
	Seat 1	Seat 2	Seat 3	Seats $i = 4, ..., s$
Points	$A_1 : 100$	$A_2 : 100$		$A_i = 500$
Points	$B_1 : 45$	$B_2 : 50$	$H_{R_3}^k : 10$	$B_i = 1$
Points	$C_1 : 0$	$C_2 : 0$	$f_3(R_3) : 5 + \varepsilon$	$C_i = 0$

sider the seat profiles (R_1, R_2). We have $f(R_1, R_2) = (f_1(R_1), f_2(R_2)) = (A_1, f_2(R_2)$, although all voters may prefer $(B_1, H_{R_{21}})$, making f inefficient. Therefore, f_1 cannot be a dictatorship for voter 1. Similarly, f_1 cannot be a dictatorship for any player, and neither can f_2.

First suppose that there are at least three voters (i.e., $n \geq 3$).

W.l.o.g., suppose that seat 1 is contested by at least three candidates.

i) Let R_1 be:

Voter 1	Voter 2	$\cdots$	Voter $n-2$	Voter $n-1$	Voter n
A_1	A_1	$\cdots$	A_1	B_1	C_1
C_1	C_1	$\cdots$	C_1	A_1	B_1
B_1	B_1	$\cdots$	B_1	C_1	A_1
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$

By efficiency $f_1(R_1)$ is either A_1 or B_1 or C_1.

We now establish that $f_1(R_1) \neq B_1$. Suppose instead that $f_1(R_1) = B_1$. Since f_2 is not a dictatorship for any player, there exists a preference profile P_2 for seat 2 such that $f_2(P_2) \neq H_{P_{2(n-1)}}$. We have $f(R_1, P_2) = (B_1, f_2(P_2))$ but the rankings (R_1, P_2) are consistent with everyone preferring $\left(C_1, H_{P_2(n-1)}\right)$, making f inefficient. Thus, we must have $f_1(R_1) \neq B_1$. Similarly, $f_1(R_1) \neq C_1$, and we conclude that $f_1(R_1) = A_1$.

ii) Now let P_2 be a profile for seat 2 in which players 1 through $n-2$ all rank, say, A_2 first. Suppose that $f_2(P_2) \neq A_2$. We have $f(R_1, P_2) = (A_1, f_2(P_2))$, but all voters may well prefer (B_1, A_2). We conclude that if the first $n-2$ voters agree on their preferred candidate, f_2 must select it.

iii) We proceed inductively. Assume that for $2 \leq j \leq n-2$, when the first $n-j$ voters agree on their preferred candidate for seat 2, f_2 selects it. We now show that the same holds true for $(j+1)$.

Define R_1^j :

Voter 1	Voter 2	$\cdots$	Voter $n-j-1$	Voter $n-j$	Voter $n-j+1$	$\cdots$	Voter n
A_1	A_1	$\cdots$	A_1	B_1	C_1	$\cdots$	C_1
C_1	C_1	$\cdots$	C_1	A_1	B_1	$\cdots$	B_1
B_1	B_1	$\cdots$	B_1	C_1	A_1	$\cdots$	A_1
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$

By efficiency $f_1(R_1)$ is either A_1 or B_1 or C_1. An argument similar to that in i) above shows that $f_1\left(R_1^j\right) \neq B_1$. Suppose that $f_1\left(R_1^j\right) = C_1$, and consider

a profile R_2^j for seat 2 in which voters 1 through $n-j$ rank A_2 first, while voters $n-j+1$ through n rank B_2 first. From the inductive assumption, $f\left(R_2^j\right)=A_2$. But then $f\left(R_1^j,R_2^j\right)=(C_1,A_2)$ although everyone may prefer (A_1,B_2). Therefore, $f_1\left(R_1^j\right)$ does not equal C_1 either, and so $f_1\left(R_1^j\right)=A_1$. Now let P_2 be a profile for seat 2 in which players 1 through $n-(j+1)$ all rank, say, A_2 first. Suppose that $f_2(P_2)\neq A_2$. We have $f(R_1,P_2)=(A_1,f_2(P_2))$, but all voters may well prefer (B_1,A_2). We conclude that if the first $n-(j+1)$ voters agree on their preferred candidate, f_2 must select it, thus establishing the inductive step.

When $j=n-2$, we have that f_2 is a dictatorship for Voter 1, a contradiction.

Finally, suppose that there are two voters (i.e, $n=2$). Since f_i is not a dictatorship, there exists a profile R_1 such that $f_1(R_1)\neq H_{R_{11}}$, and a profile R_2 such that $f_2(R_2)\neq H_{R_{22}}$. But then f is not efficient since both voters may prefer $(H_{R_{11}},H_{R_{22}})$ to $f(R_1,R_2)$. ■

Proof of Proposition 1. Let $\mathcal{A}=(A_1,...,A_s)$ and $\mathcal{B}=(B_1,...,B_s)$ be two sets of positions with $A_i,B_i\in C_i$, $A_i\neq B_i$ for all i, and let $\mathbb{A}^{AB}$ be the 2^s assemblies that can be formed from A and B. From Theorem 1 there exists a profile of assembly rankings L^{AB} over $\mathbb{A}^{AB}$ and a $C\in\mathbb{A}^{AB}$ such that $C\succ_{L_j^{AB}} f\left(R\left(L^{AB}\right)\right)\equiv D$ for all $j\in N$. We now extend the rankings L^{AB} to rankings over $\mathbb{C}_1\times...\times\mathbb{C}_s$. For each seat i, let T_i be the set of voters who prefer A_i to B_i, according to L^{AB}. Formally, $T_i=\left\{j\in N: A_i\succ_{R_{ij}\left(L^{AB}\right)} B_i\right\}$. Since preferences are assumed strict, for $j\in N\backslash T_i$, $B_i\succ_{R_{ij}\left(L^{AB}\right)} A_i$. Let $R\in\mathbb{R}^{n,s}$ be a candidate seat profile such that for each seat i, for each voter $j\in T_i$, candidate A_i is the top-ranked candidate and B_i is the second-ranked candidate, while for each $j\in N\backslash T_i$, candidate B_i is the top-ranked candidate and A_i is the second-ranked candidate. Let L be a profile of assembly rankings over $\mathbb{A}$ which is consistent with R and such that $C\succ_{L_i} D$ for all $i\in N$.

We note the following:

a) Given any two candidates for a seat i, since f_i is monotonic, for any voter either it is dominated to vote for his least preferred candidate, or his vote never matters. Therefore, w.l.o.g. we can assume that in an undominated equilibrium each player votes for his preferred candidate.

b) Monotonicity and voter sovereignty imply that f is seat-by-seat efficient.

c) In equilibrium, each candidate must have a 50% chance of winning, since deviating to the other candidate's position guarantees this much.

For each seat i, no position $X_i \notin (A_i, B_i)$ can be chosen with strictly positive probability, since from a) and b) this position will lose to either A_i or B_i. Therefore, suppose that for every i, each candidate is at position A_i or B_i. If there is one candidate at each position, by construction, and from c), candidate D_i is elected. Therefore, both candidates will in fact enter at D_i, and D_i is elected. ■

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